Revista Alma Mater
e-ISSN: 3060-9917
2026 | Revista Alma Mater, 1(1), 1–15
www.revistaalmamater.org

Tensorial Formulation of the Quantum Harmonic Oscillator in Curvilinear Coordinates

Formulación tensorial del oscilador armónico cuántico en coordenadas curvilíneas

Juan Carlos Marine*
Institute of Physics, Faculty of Sciences, Universidad Autónoma de Santo Domingo, Dominican Republic
jmarine42@uasd.edu.do
ORCID: 0009-0004-5557-9387

Juan Toribio Milane
Institute of Physics & Institute of Mathematics, Faculty of Sciences, Universidad Autónoma de Santo Domingo, Dominican Republic
jtoribio34@uasd.edu.do
ORCID: 0000-0002-0782-1827

Kelvin Antonio Florimón de Jesús
Ministry of Education of the Dominican Republic
kelvin.florimon@docente.edu.do
ORCID: 0009-0009-8312-9654

José Miguel Sánchez Gómez
School of Mathematics and Physics, Universidad Tecnológica del Cibao Oriental, Dominican Republic
josem.sanchez@uteco.edu.do
ORCID: 0009-0006-0673-7136

Received: 20/09/2025 Accepted: 28/03/2026 Published: 14/04/2026
*Corresponding author: Juan Carlos Marine; jmarine42@uasd.edu.do

DOI: https://doi.org/10.61780/ag1fae81

Cómo citar: Marine, J. C., Toribio Milane, J., Florimón de Jesús, K. A. y Sánchez Gómez, J. M. (2026). Tensorial Formulation of the Quantum Harmonic Oscillator in Curvilinear Coordinates. *Revista Alma Mater*, *1*(1), 1–15. https://doi.org/10.61780/ag1fae81

Abstract

This article presents a tensorial formulation for the analysis of the three-dimensional isotropic quantum harmonic oscillator in curvilinear coordinates. Starting from the metric tensor and its associated geometric structures, the line element, scale factors, and invariant surface and volume elements are derived, together with the Laplace–Beltrami operator in orthogonal systems. Within this geometric framework, the stationary Schrödinger equation is solved in Cartesian, cylindrical, and spherical coordinates.

The separation of variables leads naturally to families of orthogonal polynomials, namely Hermite and associated Laguerre polynomials, whose orthogonality is determined by the corresponding Riemannian measures. The resulting spectrum, $E_N = \hbar\omega\,(N + \frac{3}{2})$, exhibits the characteristic degeneracy $g_N = \frac{(N+1)(N+2)}{2}$, reflecting the isotropy of the potential.

Regularity and self-adjointness conditions are analyzed to ensure the physical admissibility of the eigenfunctions. In addition, the metric and Laplacian in cylindrical elliptic coordinates are introduced, providing a basis for future studies on separability and anisotropic extensions.

Keywords: Quantum harmonic oscillator, tensorial formalism, curvilinear coordinates, Schrödinger equation, orthogonal polynomials.

Resumen

Resumen

Este artículo presenta una formulación tensorial para el análisis del oscilador armónico cuántico isotrópico tridimensional en coordenadas curvilíneas. A partir del tensor métrico y de sus estructuras geométricas asociadas, se derivan el elemento de línea, los factores de escala y los elementos invariantes de superficie y volumen, junto con el operador de Laplace–Beltrami en sistemas ortogonales. Dentro de este marco geométrico, se resuelve la ecuación de Schrödinger estacionaria en coordenadas cartesianas, cilíndricas y esféricas.

La separación de variables conduce de manera natural a familias de polinomios ortogonales, a saber, los polinomios de Hermite y de Laguerre asociados, cuya ortogonalidad está determinada por las correspondientes medidas riemannianas. El espectro resultante, $E_N = \hbar\omega\,(N + \frac{3}{2})$, presenta la degeneración característica $g_N = \frac{(N+1)(N+2)}{2}$, reflejo de la isotropía del potencial.

Se analizan condiciones de regularidad y autoadjunción para garantizar la admisibilidad física de las autofunciones. Además, se introducen la métrica y el operador laplaciano en coordenadas elípticas cilíndricas, proporcionando una base para estudios futuros sobre separabilidad y extensiones anisotrópicas.

Palabras clave: Oscilador armónico cuántico, formalismo tensorial, coordenadas curvilíneas, ecuación de Schrödinger, polinomios ortogonales.

1 Introduction

The three-dimensional isotropic quantum harmonic oscillator is a paradigmatic exactly solvable system that admits separable representations in multiple coordinate systems. This multiplicity raises a structural question: how these distinct formulations encode the same spectral and physical content.

In standard treatments, the oscillator is typically solved independently in each coordinate system, while the geometric structures underlying their equivalence—namely the metric tensor, the invariant measure, the scale factors, and the Laplace–Beltrami operator—are introduced in a coordinate-dependent manner. As a result, the intrinsic relation between geometry, separability, normalization, and admissibility remains partially obscured. The aim of this work is therefore not to rederive the well-known spectrum, but to provide a unified tensorial formulation that explains its coordinate-independent origin.

A related coordinate-wise analysis has been previously considered in the literature (see Ref. [8]), although without a covariant tensorial organization. The present work reformulates and extends that approach within a differential-geometric framework, without reproducing prior derivations.

The main objectives of this article are as follows. We derive, in a unified manner, the geometric structures associated with orthogonal curvilinear coordinates in $\mathbb{R}^3$, including the metric tensor, scale factors, and invariant surface and volume elements. The kinetic term of the Schrödinger Hamiltonian is then expressed through the Laplace–Beltrami operator, ensuring manifest coordinate covariance. Within this framework, the stationary isotropic oscillator is solved in Cartesian, cylindrical, and spherical coordinates, and the origin of the common spectrum and its degeneracy is clarified. As an extension, the metric and Laplacian in cylindrical elliptic coordinates are also introduced.

From a methodological standpoint, we consider smooth local transformations $x = f(q)$ and employ the induced metric g_{ij} to derive the line element, scale factors, invariant measures, and the tensorial form of the Laplacian [4, 11]. These structures are then specialized to orthogonal coordinate systems of physical interest. In each case, separation of variables reduces the stationary Schrödinger equation to ordinary differential equations subject to regularity, square-integrability, and hermiticity conditions imposed by the corresponding measures. Hermite and associated Laguerre families thus emerge naturally as consequences of a single geometric framework.

The main contribution of this work is the systematic identification of the Laplace–Beltrami operator as the universal kinetic operator of the problem, together with the interpretation of smooth coordinate transformations as unitary isomorphisms between the corresponding Hilbert spaces. Regularity conditions at coordinate singularities are explicitly related to the effective self-adjoint domain of the Hamiltonian, while the appearance of special-function families is understood as a Sturm–Liouville consequence of the underlying Riemannian measure. In this way, the oscillator is interpreted as a single geometric object whose different coordinate realizations correspond to equivalent representations of the same quantum system.

The manuscript is organized as follows. Section 2 develops the geometric foundations of orthogonal curvilinear coordinates, including the metric tensor, scale factors, invariant measures, and the tensorial Laplacian. Section 3 applies this framework to the isotropic quantum harmonic oscillator and presents the separated solutions in Cartesian, cylindrical, and spherical coordinates. Section 4 discusses the geometric interpretation of covariance, degeneracy, regularity, and separability, and Section 5 summarizes the main results and scope of the work.

2 Geometric Structure of Curvilinear Coordinates

2.1 Orthogonal Coordinates and the Metric Tensor

Let $P \in \mathbb{R}^3$. Consider two local curvilinear coordinate systems representing the same point P [13, p. 166], namely (q^1, q^2, q^3) and $(\overline{q}^1, \overline{q}^2, \overline{q}^3)$.

Assume that the transformation between these systems is smooth and locally invertible, i.e., a *local diffeomorphism*. This ensures a unique representation of the point and excludes foldings in a neighborhood of P.

Suppose that the coordinate transformation is given by three functionally independent relations with continuous derivatives,

$$\begin{cases} \overline{q}^1 = \overline{q}^1\left(q^1, q^2, q^3\right), \\ \overline{q}^2 = \overline{q}^2\left(q^1, q^2, q^3\right), \\ \overline{q}^3 = \overline{q}^3\left(q^1, q^2, q^3\right), \end{cases} \tag{1}$$

with inverse relations

$$q^k = q^k\left(\overline{q}^1, \overline{q}^2, \overline{q}^3\right), \qquad k = 1, 2, 3. \tag{2}$$

Equations (1)–(2) define the coordinate transformation between the two systems.

Having established the coordinate transformation, we now examine how infinitesimal displacements transform under this change of variables.

2.2 Total Differential

The transformation properties of infinitesimal displacements follow from the coordinate transformation introduced above. The total differential of (1) is

$$d\overline{q}^k = \frac{\partial \overline{q}^k}{\partial q^l}\, dq^l, \qquad k = 1, 2, 3. \tag{3}$$

In $\mathbb{R}^3$, this yields explicitly $d\overline{q}^1 = \frac{\partial \overline{q}^1}{\partial q^l}\, dq^l$, $d\overline{q}^2 = \frac{\partial \overline{q}^2}{\partial q^l}\, dq^l$, and $d\overline{q}^3 = \frac{\partial \overline{q}^3}{\partial q^l}\, dq^l$.

In matrix form,

$$\begin{pmatrix} d\overline{q}^1 \\ d\overline{q}^2 \\ d\overline{q}^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial \overline{q}^1}{\partial q^1} & \frac{\partial \overline{q}^1}{\partial q^2} & \frac{\partial \overline{q}^1}{\partial q^3} \\ \frac{\partial \overline{q}^2}{\partial q^1} & \frac{\partial \overline{q}^2}{\partial q^2} & \frac{\partial \overline{q}^2}{\partial q^3} \\ \frac{\partial \overline{q}^3}{\partial q^1} & \frac{\partial \overline{q}^3}{\partial q^2} & \frac{\partial \overline{q}^3}{\partial q^3} \end{pmatrix}}_{\dfrac{\partial \overline{q}}{\partial q}} \begin{pmatrix} dq^1 \\ dq^2 \\ dq^3 \end{pmatrix}.$$

The inverse relations take the form

$$dq^l = \frac{\partial q^l}{\partial \overline{q}^j}\, d\overline{q}^j. \tag{4}$$

Substituting (4) into (3) and relabeling $k \to i$ yields

$$d\overline{q}^i = \frac{\partial \overline{q}^i}{\partial q^l}\, \frac{\partial q^l}{\partial \overline{q}^j}\, dq^j, \tag{5}$$

where $\frac{\partial \overline{q}^i}{\partial q^l} \frac{\partial q^l}{\partial \overline{q}^j} = \delta^i{}_j$ [6, p. 41], i.e., the Kronecker identity.

Equations (3)–(5) show that the same infinitesimal displacement of a point P toward a neighboring point Q (vector $\overrightarrow{PQ}$) has components dq^k in one system and $d\overline{q}^k$ in the other; both correspond to the same geometric vector expressed in different coordinate bases.

The independence of (1) and (2) holds if and only if the Jacobian matrix $J\left(\frac{q}{\overline{q}}\right)$ is nonzero and finite at P, where

$$J\left(\frac{q}{\overline{q}}\right) = \begin{pmatrix} \frac{\partial q^1}{\partial \overline{q}^1} & \frac{\partial q^1}{\partial \overline{q}^2} & \frac{\partial q^1}{\partial \overline{q}^3} \\ \frac{\partial q^2}{\partial \overline{q}^1} & \frac{\partial q^2}{\partial \overline{q}^2} & \frac{\partial q^2}{\partial \overline{q}^3} \\ \frac{\partial q^3}{\partial \overline{q}^1} & \frac{\partial q^3}{\partial \overline{q}^2} & \frac{\partial q^3}{\partial \overline{q}^3} \end{pmatrix}. \tag{6}$$

IIf the Jacobian in (6) is nonvanishing, then the inverse Jacobian $\overline{J}\left(\frac{\overline{q}}{q}\right)$ exists and satisfies $J\,\overline{J} = I_3$.

The condition that the Jacobian in (6) is nonzero excludes degenerate coordinate systems, ensures local uniqueness of the representation, and guarantees the correct transformation of densities and measures, such as the volume element.

2.3 Tangent and Dual Basis; the Metric Tensor

Let $\vec{e}_i = \partial\vec{r}/\partial q^i$ denote the *tangent (covariant) basis*, and let $\vec{e}^{\,i}$ denote its dual (contravariant) basis.

The components of the metric tensor at P are defined by [11, p. 612]

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j = \frac{\partial x^k}{\partial q^i}\, \frac{\partial x^k}{\partial q^j}. \tag{7}$$

The contravariant and mixed components are given by $g^{ij} = \vec{e}^{\,i} \cdot \vec{e}^{\,j} = \frac{\partial q^i}{\partial x^k} \frac{\partial q^j}{\partial x^k}$ and $g^i{}_j = \vec{e}^{\,i} \cdot \vec{e}_j = \frac{\partial q^i}{\partial x^k} \frac{\partial x^k}{\partial q^j} = \delta^i{}_j$.

The metric coefficients g_{ij} determine lengths and angles through $ds^2 = g_{ij}\, dq^i dq^j$, while g^{ij} denotes the inverse matrix of $[g_{ij}]$, used to raise indices and establish the duality between covariant and contravariant components. In this way, the geometric structure is independent of the choice of coordinates.

The duality relations $g_{ij}\,\vec{e}^{\,j} = \vec{e}_i$ and $g^{ij}\,\vec{e}_j = \vec{e}^{\,i}$ hold [1, p. 154].

For an *orthogonal* coordinate system, the metric tensor is diagonal, i.e., $g_{ij} = 0$ for $i \neq j$, and consequently $g^{ij} = 0$ for $i \neq j$. Moreover, it holds [7, p. 577]

$$[g_{ij}]\,[g^{jk}] = \delta_i^k,$$

which implies that $[g^{ij}] = [g_{ij}]^{-1}$.

2.4 Arc Differential and Scale Factors

To relate the arc differential to the metric tensor, recall the total differential $d\vec{r}$ [8, p. 16]. If $\vec{e}_i := \partial\vec{r}/\partial q^i$ denotes the covariant basis, then

$$d\vec{r} = \frac{\partial x^k}{\partial q^i}\,dq^i\,\hat{\mathbf{e}}_k = \vec{e}_i\,dq^i.$$

The arc differential satisfies

$$(ds)^2 = d\vec{r}\cdot d\vec{r} = (\vec{e}_i\cdot\vec{e}_j)\,dq^i dq^j.$$

Using (7), $g_{ij} = \vec{e}_i\cdot\vec{e}_j$, we obtain

$$(ds)^2 = g_{ij}\,dq^i dq^j. \tag{8}$$

In an *orthogonal* coordinate system [9], the metric tensor is diagonal, $g_{ij} = \delta_{ij}\,h_i^2$, where the scale factors are defined as [8, p. 19] $h_i := \|\partial\vec{r}/\partial q^i\| = \sqrt{g_{ii}}$. Thus, the line element reduces to $(ds)^2 = \sum_{i=1}^3 h_i^2\,(dq^i)^2$.

From the definition of g_{ii}, it follows that

$$h_i^2 = \sum_{k=1}^{3}\left(\frac{\partial x^k}{\partial q^i}\right)^2 = g_{ii}, \qquad \text{(no sum on } i\text{)}. \tag{9}$$

Therefore, the h_i are the scale factors of the orthogonal curvilinear system. In particular, the arc length component along q^i is

$$ds_i = \sqrt{g_{ii}}\,dq^i, \qquad \text{(no sum on } i\text{)}. \tag{10}$$

For the contravariant metric,

$$g^{ij} = \frac{\partial q^i}{\partial x^k}\frac{\partial q^j}{\partial x^k}, \qquad g^{ij} = 0\ (i\neq j), \quad g^{ii} = \frac{1}{h_i^2}.$$

The determinant of the metric satisfies

$$g = \det[g_{ij}] = \prod_{i=1}^{3} h_i^2, \qquad \sqrt{g} = \Big|\det(\partial x/\partial q)\Big|.$$

For the contravariant metric [12, p. 31],

$$\overline{g} = \det[g^{ij}] = \prod_{i=1}^{3} \frac{1}{h_i^2}.$$

Finally, in orthogonal systems the metric determinant and the Jacobian J of the transformation $x = x(q)$ satisfy

$$g = \prod_{i=1}^{3} h_i^2 = J^2.$$

2.5 Surface and Volume Elements

We now derive the invariant surface and volume elements associated with the curvilinear coordinate system (q^1, q^2, q^3).

The surface element corresponding to the level set $q^i = \text{const}$ is given by

$$dS_{q^i} = \sqrt{g\, g^{ii}}\, dq^j\, dq^k, \qquad (\text{no sum on } i,\ \{i,j,k\} \text{ cyclic}). \tag{11}$$

In an *orthogonal* coordinate system, using $g^{ii} = 1/h_i^2$ and $\sqrt{g} = h_1 h_2 h_3$, this reduces to

$$dS_{q^i} = \frac{h_1 h_2 h_3}{h_i}\, dq^j\, dq^k = h_j h_k\, dq^j\, dq^k.$$

The volume element is given by

$$dV = \sqrt{g}\, dq^1\, dq^2\, dq^3. \tag{12}$$

In orthogonal systems [12, p. 39], this expression becomes

$$dV = h_1 h_2 h_3\, dq^1 dq^2 dq^3.$$

Moreover, the relation between the metric determinant and the Jacobian of the transformation $x = x(q)$ implies that

$$dV = |J|\, dq^1 dq^2 dq^3 = \sqrt{g}\, dq^1 dq^2 dq^3,$$

which ensures the invariance of the volume measure under smooth coordinate transformations [11, p. 616].

2.6 The Laplacian Operator in Tensorial Form

Let φ be a scalar field defined on a three-dimensional space endowed with a metric g_{ij}. The Laplacian, in its invariant tensorial form (Laplace–Beltrami operator), is given by [4, eq. 11.29, p. 305]

$$\nabla^2 \varphi = \frac{1}{\sqrt{g}}\, \partial_i \Big(\sqrt{g}\, g^{ij}\, \partial_j \varphi \Big). \tag{13}$$

This expression makes explicit that the Laplacian depends solely on the metric structure and is therefore invariant under smooth coordinate transformations.

Examples. In standard orthogonal coordinate systems, (13) reduces to the following familiar forms:

- Cartesian (x, y, z):
$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}.$$
- Cylindrical (r, θ, z):
$$\nabla^2\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2}.$$
- Spherical (r, θ, ϕ):
$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}.$$
- Cylindrical elliptic (u, v, z):
$$\nabla^2\varphi = \frac{1}{a^2\left(\sinh^2 u + \sin^2 v\right)}\left(\frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2}\right) + \frac{\partial^2\varphi}{\partial z^2}. \tag{14}$$

2.7 Some Differential Operators in Tensorial Form

Gradient. Let $\varphi = \varphi(q^1, q^2, q^3)$ be a scalar field. Its gradient is given by
$$\nabla\varphi = \vec{e}^{\,i}\,\partial_i\varphi.$$

Divergence. For a vector field $A = A^i\vec{e}_i$, the divergence reads
$$\nabla\cdot A = \frac{1}{\sqrt{g}}\,\partial_i\!\left(\sqrt{g}\,A^i\right). \tag{15}$$

Curl. For a vector field $A = A^i\vec{e}_i$, the curl can be expressed in contravariant components as
$$(\nabla\times A)^k = \frac{1}{\sqrt{g}}\,\epsilon^{k}_{\ ij}\,\partial_i A^j,$$
where $\epsilon^{k}_{\ ij}$ denotes the Levi–Civita symbol.

In orthogonal coordinate systems, using the scale factors h_i, this expression is equivalent to
$$(\nabla\times A)^1 = \frac{1}{h_2h_3}\Big[\partial_2(h_3A^3) - \partial_3(h_2A^2)\Big],$$
$$(\nabla\times A)^2 = \frac{1}{h_1h_3}\Big[\partial_3(h_1A^1) - \partial_1(h_3A^3)\Big],$$

$$(\nabla \times A)^3 = \frac{1}{h_1 h_2}\Big[\partial_1(h_2 A^2) - \partial_2(h_1 A^1)\Big].$$

A compact representation is given by [11, p. 632]

$$(\nabla \times A)^k = \frac{\epsilon_{ijk}\, h_k}{\sqrt{g}}\, \partial_i A^j.$$

3 Results

3.1 Tensorial form of the stationary problem

We consider the stationary isotropic three-dimensional quantum harmonic oscillator. Introducing the parameter $\alpha := \frac{m\omega}{\hbar}$, the characteristic oscillator length is given by $a_{\mathrm{ho}} = \alpha^{-1/2} = \sqrt{\frac{\hbar}{m\omega}}$.

The Hamiltonian reads

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \tfrac{1}{2} m\omega^2 r^2, \tag{16}$$

and the stationary Schrödinger equation is

$$\hat{H}\,\Psi(\mathbf{r}) = E\,\Psi(\mathbf{r}). \tag{17}$$

Using the Laplace–Beltrami operator in the form (13), the stationary equation in a general orthogonal curvilinear coordinate system becomes

$$-\frac{\hbar^2}{2m}\,\frac{1}{\sqrt{g}}\,\partial_i\Big(\sqrt{g}\,g^{ij}\partial_j\Psi\Big) + \frac{1}{2}m\omega^2 r^2\,\Psi = E\,\Psi. \tag{18}$$

Equation (18) shows that the kinetic term is entirely determined by the metric structure, so that the same quantum system admits equivalent representations in different coordinate systems. In the cases considered below, separability follows from the orthogonality of the metric and the symmetry of the potential.

3.2 Cartesian coordinates

In Cartesian coordinates, we adopt the separable ansatz

$$\Psi(\mathbf{r}) = X(x)\,Y(y)\,Z(z).$$

The Laplacian takes the form $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$, so that (17) separates into three independent one-dimensional oscillator equations.

Introducing the dimensionless variables $\xi_i = \sqrt{\alpha}\,q_i$, with $q_i \in \{x, y, z\}$, each equation reduces to the Hermite differential equation. The normalized one-dimensional solutions are

$$X_{n_x}(x) = \frac{\alpha^{1/4}}{\pi^{1/4}\sqrt{2^{n_x} n_x!}}\,H_{n_x}(\sqrt{\alpha}\,x)\,e^{-\alpha x^2/2}, \tag{19}$$

and analogously for $Y_{n_y}(y)$ and $Z_{n_z}(z)$. Hence, $\Psi_{n_x n_y n_z}(\mathbf{r}) = X_{n_x}(x)\, Y_{n_y}(y)\, Z_{n_z}(z)$, with energy spectrum $E_{n_x n_y n_z} = \hbar\omega\Big(n_x + n_y + n_z + \frac{3}{2}\Big)$. Defining $N := n_x + n_y + n_z$, we obtain $E_N = \hbar\omega\Big(N + \frac{3}{2}\Big)$ with $N \in \mathbb{N}_0$.

The Gaussian factor sets the characteristic length scale a_{ho}, while the Hermite polynomials determine the nodal structure along each coordinate axis. The degeneracy at fixed N is purely combinatorial and given by $g_N = \binom{N+2}{2} = \frac{(N+1)(N+2)}{2}$.

3.3 Cylindrical coordinates

In cylindrical coordinates, we adopt the ansatz

$$\Psi(\rho, \phi, z) = R(\rho)\, e^{i\mu\phi}\, Z(z), \qquad \mu \in \mathbb{Z}.$$

The factor $e^{i\mu\phi}$ enforces single-valuedness in the azimuthal variable and identifies μ as the eigenvalue of $\hat{L}_z/\hbar$. The longitudinal factor $Z(z)$ satisfies the same one-dimensional oscillator equation as in Cartesian coordinates. The radial function obeys

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \frac{\mu^2}{\rho^2}R - \alpha^2\rho^2 R + \lambda R = 0, \tag{20}$$

where λ is a separation constant.

The parameter $\alpha = m\omega/\hbar$ is kept fixed, while the dimensionless variable $t = \alpha\rho^2$ is introduced to reduce (20) to the associated Laguerre form.

Imposing regularity at $\rho = 0$ and square-integrability with respect to the measure $\rho\, d\rho$, the radial solution takes the form

$$R_{n_\rho \mu}(\rho) = \mathcal{N}_\rho\, \rho^{|\mu|} e^{-\alpha\rho^2/2}\, L_{n_\rho}^{|\mu|}(\alpha\rho^2), \qquad n_\rho \in \mathbb{N}_0.$$

The term μ^2/ρ^2 acts as a centrifugal barrier, shifting the radial density away from the axis as $|\mu|$ increases. The energy spectrum is

$$E = \hbar\omega\Big(2n_\rho + |\mu| + n_z + \tfrac{3}{2}\Big) = \hbar\omega\Big(N + \tfrac{3}{2}\Big), \qquad N = 2n_\rho + |\mu| + n_z.$$

3.4 Spherical coordinates

In spherical coordinates, we adopt the representation

$$\Psi(r, \theta, \phi) = R_{n\ell}(r)\, Y_\ell^m(\theta, \phi),$$

with $\ell \in \mathbb{N}_0$ and $m = -\ell, \ldots, \ell$. Introducing $u(r) := r\, R(r)$, the radial equation corresponding to (17) takes the form

$$u''(r) + \left[\frac{2m}{\hbar^2}\left(E - \tfrac{1}{2}m\omega^2 r^2\right) - \frac{\ell(\ell+1)}{r^2}\right] u(r) = 0.$$

The effective potential is

$$V_{\text{eff}}(r) = \tfrac{1}{2}m\omega^2 r^2 + \frac{\hbar^2 \ell(\ell+1)}{2mr^2},$$

where the second term represents the centrifugal barrier. With $z = \sqrt{\alpha}\, r$ and the ansatz

$$u(z) = z^{\ell+1} e^{-z^2/2} F(z^2),$$

one obtains a confluent hypergeometric equation. Normalizability on $[0, \infty)$ enforces polynomial truncation, yielding associated Laguerre functions:

$$R_{n\ell}(r) = \mathcal{N}_r\, r^\ell e^{-\alpha r^2/2}\, L_{n_r}^{\ell+1/2}(\alpha r^2), \qquad E = \hbar\omega\Big(2n_r + \ell + \tfrac{3}{2}\Big)\,. \tag{21}$$

Defining $N = 2n_r + \ell$, we recover

$$E_N = \hbar\omega\Big(N + \tfrac{3}{2}\Big)\,.$$

For fixed N, the allowed values of ℓ are

$$\ell = N, N-2, \ldots, \qquad \ell = \begin{cases} 0, & N \text{ even}, \\ 1, & N \text{ odd}. \end{cases}$$

Each ℓ contributes with multiplicity $2\ell + 1$, and the resulting sum reproduces the degeneracy g_N obtained in Cartesian coordinates.

3.5 Common spectrum and degeneracy

The three coordinate representations yield distinct families of separated solutions, yet they all lead to the same isotropic spectrum,

$$E_N = \hbar\omega\Big(N + \tfrac{3}{2}\Big)\,, \qquad g_N = \frac{(N+1)(N+2)}{2}.$$

In Cartesian coordinates, the degeneracy corresponds to the number of partitions of $N = n_x + n_y + n_z$ into three nonnegative integers. In cylindrical coordinates, the same energy shell is characterized by $N = 2n_\rho + |\mu| + n_z$. In spherical coordinates, it is described by $N = 2n_r + \ell$, together with the multiplicity $2\ell + 1$ of the spherical harmonics.

The agreement among these counting schemes reflects the isotropy of the potential and is consistent with the symmetry-based interpretation of the oscillator developed by Fradkin [5].

4 Discussion

The main contribution of this work is not the derivation of a new spectral formula for the isotropic oscillator, but the identification of a unified geometric mechanism underlying its coordinate-invariant structure. Starting from the induced metric, we constructed the line element, scale factors, invariant surface and volume elements, and the Laplace–Beltrami operator (13) in a fully covariant form. Within this framework, the oscillator is naturally interpreted as a single quantum system admitting multiple equivalent coordinate representations.

A central aspect of this formulation is the role of the Riemannian measure induced by the transformation $x = f(q)$. In particular, the volume element

$$dV = \sqrt{g}\, d^3q,$$

as given in (12), defines a natural unitary mapping

$$U_f : \; L^2(\mathbb{R}^3, d^3x) \longrightarrow L^2(\mathbb{R}^3, \sqrt{g}\, d^3q), \qquad (U_f\psi)(q) = \psi\big(f(q)\big).$$

The identity

$$\int_{\mathbb{R}^3} |\psi(x)|^2\, d^3x = \int_{\mathbb{R}^3} |\psi(f(q))|^2 \sqrt{g}\, d^3q$$

shows that both the spectrum and probability densities are invariant under smooth coordinate transformations, with only the representation of the state being modified. This provides a precise Hilbert-space interpretation of coordinate covariance.

The tensorial formulation also clarifies the origin of the special-function families arising in each coordinate system. Hermite-type solutions in Cartesian coordinates, given in (19), associated Laguerre functions in cylindrical coordinates, obtained from the radial equation (20), and their spherical counterparts in (21), all arise from Sturm–Liouville problems whose weights are determined by the underlying metric. The corresponding orthogonality relations are therefore intrinsic consequences of the Riemannian measure, rather than externally imposed conditions.

Admissibility of solutions is likewise encoded in the geometric structure. Regularity at coordinate singularities, such as $\rho = 0$ and $r = 0$, excludes nonphysical branches that violate square-integrability or finite energy conditions. The divergence formula (15), together with the measure $\sqrt{g}\, d^3q$, governs the boundary terms arising under integration by parts and determines the effective self-adjoint domain of the Hamiltonian. In this sense, the selection of admissible solutions is a direct consequence of the geometric formulation.

From a broader perspective, the present treatment connects classical results on separability of orthogonal coordinate systems, originating in the work of Eisenhart on Stäckel systems [3], with symmetry-based interpretations of degeneracy, such as those developed by Fradkin [5]. Rather than replacing these approaches, the tensorial framework provides a unified setting in which metric structure, invariant measures, and separable quantum equations are derived simultaneously.

Revista Alma Mater
e-ISSN: 3060-9917
2026 | Revista Alma Mater, 1(1), 1–15
www.revistaalmamater.org

The extension to cylindrical elliptic coordinates further illustrates the flexibility of the approach. The tensorial expression of the Laplacian in (14) shows that the formalism extends naturally to nonstandard orthogonal systems. Although a complete spectral solution was not pursued in this case, the construction establishes a basis for future studies on separability and anisotropic generalizations.

In summary, the tensorial formulation developed here organizes the isotropic harmonic oscillator into a geometrically coherent framework in which covariance, separability, and spectral invariance emerge as direct consequences of the metric structure.

5 Conclusions

We have developed a coordinate-covariant tensorial formulation of the isotropic three-dimensional quantum harmonic oscillator in orthogonal curvilinear coordinates. Starting from the induced metric tensor g_{ij} in (7), we obtained the line element in (8), the scale factors in (9)–(10), the invariant surface and volume elements in (11) and (12), and the associated Riemannian measure $dV = \sqrt{g}\, d^3q$. Within this geometric framework, the kinetic term is expressed through the Laplace–Beltrami operator (13), which preserves the form of the stationary Schrödinger equation under smooth coordinate transformations.

By separation of variables in Cartesian, cylindrical, and spherical coordinates, starting from the Hamiltonian (16) and the stationary Schrödinger equation (17), we showed that the same isotropic spectrum

$$E_N = \hbar\omega\Big(N + \tfrac{3}{2}\Big)\,, \qquad g_N = \frac{(N+1)(N+2)}{2}$$

is recovered in all representations. The analysis also makes explicit how the Hermite family (19), the cylindrical radial equation (20), and the spherical radial solutions (21) arise from the Sturm–Liouville structure induced by the corresponding measures. Moreover, regularity at coordinate singularities is naturally linked to the physically admissible domain of the Hamiltonian.

The scope of this work is therefore precise. Rather than introducing new spectral results, it provides a rigorous geometric organization of known solutions, clarifies their equivalence at the level of operators and Hilbert spaces, and establishes a coherent framework connecting tensor calculus, orthogonal curvilinear coordinates, and quantum-mechanical separability.

This formulation also lays the groundwork for further developments. In particular, the extension to more general orthogonal systems, such as cylindrical elliptic coordinates, suggests a natural path toward the study of anisotropic models and more general separable potentials within the same tensorial framework.

Declarations

Authors' contributions

All authors contributed to the preparation of this manuscript. Juan Toribio Milane conceived the central idea of the work and contributed to all sections, including the analytical development of the quantum models. Juan Carlos Marine led the formulation of the tensorial representation and the overall geometric framework. Kelvin Antonio Florimón de Jesús derived the Laplacians in curvilinear coordinates and their applications. José Miguel Sánchez Gómez contributed to the analysis of eigenfunctions and the spectral structure. All authors participated in the discussion of the results and approved the final version of the manuscript.

How to cite this article

Marine, J. C., Toribio Milane, J., Florimón de Jesús, K. A., & Sánchez Gómez, J. M. (2026). *Tensorial formulation of the quantum harmonic oscillator in curvilinear coordinates / Formulación tensorial del oscilador armónico cuántico en coordenadas curvilíneas*. *Revista Alma Mater*, [volume(issue)], [pp–pp]. https://doi.org/[DOI]

Funding

The research of Juan Toribio Milane was partially supported by the National Fund for Innovation and Scientific and Technological Development (FONDOCYT), Dominican Republic, under project number 2024-2-1D2-0791.

Competing interests

The authors declare that they have no competing interests.

Generative AI use

No generative AI tools were used in the preparation of this manuscript.

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