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# Methodological Guide to Solving Optimization Problems in Single-Variable Differential Calculus Incorporating Technological Resources

## *Guía metodológica para la resolución de problemas de optimización en cálculo diferencial de una variable con recursos tecnológicos*

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## RESUMEN

Este artículo propone una guía metodológica para la enseñanza y evaluación de problemas de optimización en cálculo diferencial de una variable. La guía está organizada en cinco fases —comprensión del problema; formulación matemática; diseño de la estrategia; implementación y resultados; y verificación y resolución— y, para cada fase, especifica criterios observables e indicadores de logro que articulan el conocimiento conceptual, procedimental y metacognitivo. Estos indicadores proporcionan evidencia evaluable para la retroalimentación formativa y la construcción de rúbricas. El marco integra el modelado cartesiano y las heurísticas de tipo Pólya dentro de un enfoque basado en competencias (modelado, argumentación, comunicación y uso crítico de herramientas digitales), fomentando la autonomía, la justificación matemática rigurosa y la validación. Las tecnologías de la información y la comunicación (por ejemplo, GeoGebra, Desmos, Wolfram Alpha, wxMaxima, Symbolab) se incorporan para apoyar la visualización, el cálculo simbólico y la verificación sin subordinar el razonamiento al software. El resultado es un andamiaje replicable y transferible con criterios de desempeño claros que fortalecen la

## *ABSTRACT*

*This article presents a methodological guide for the instruction and assessment of optimization problems in single-variable differential calculus. The guide is structured into five clearly defined phases—problem comprehension; mathematical formulation; strategy design; implementation and results; and verification and problem resolution—and, for each phase, it specifies observable criteria and indicators of achievement that connect conceptual, procedural, and metacognitive knowledge. These indicators provide measurable evidence for formative feedback and rubric development. The framework integrates Cartesian modeling and Pólya-style heuristics within a competency-based approach (modeling, argumentation, communication, and the critical use of digital tools), promoting autonomy, rigorous mathematical justification, and validation. Information and communication technologies (e.g., GeoGebra, Desmos, Wolfram Alpha, wxMaxima, Symbolab) are incorporated to support visualization, symbolic computation, and verification without subordinating reasoning to software. The result is a replicable and transferable scaffold, with clearly defined performance criteria, that enhances coherence among modeling,* coherencia entre el modelado, el análisis y la evaluación en la enseñanza del cálculo.

**Palabras clave:** optimización; educación matemática; resolución de problemas; competencias; TIC en la enseñanza de la matemática

*analysis, and evaluation in calculus instruction.*

***Keywords:*** *optimization; mathematics education; problem solving; competencies; ICT in mathematics education*

## 1. INTRODUCTION

From the standpoint of pure mathematics, the field may be defined as the discipline that, through abstraction and proof, studies structures, relations, and properties within axiomatic systems (e.g., numbers, functions, spaces, and operators). From an applied perspective, mathematics functions as both a language and a conceptual tool for modeling, quantifying, and optimizing phenomena across the natural and social sciences, engineering, and data-intensive domains. According to López-Bermúdez, Hidalgo-Hidalgo, Medrano-Freire, and Barba-López (2024), the growing complexity of contemporary problems has intensified the need for data-driven solutions, placing applied mathematics—with its tools for data analysis, optimization, and modeling—in a pivotal position to address these challenges. Disciplines such as artificial intelligence, machine learning, and big data rely heavily on mathematics, particularly in the development of advanced algorithms for information analysis.

In the teaching–learning process of mathematics, three major approaches are commonly distinguished: the idealist or Platonic, the constructivist, and the empiricist. Godino, Batanero, and Font (2003) argue that students must first acquire the fundamental structures of mathematics in an axiomatic manner—that is, a solid grounding in foundational concepts must be assimilated and consolidated before application becomes feasible. This view is typically referred to as the idealist–Platonic stance. In line with this perspective, Gómez (2022) maintains that meaningful application of mathematics—except in trivial cases—presupposes a robust command of fundamentals; proponents view mathematics as an autonomous discipline that may develop independently of external applications, focusing primarily on problems internal to the field itself.

The constructivist perspective, for its part, grants greater autonomy to students and positions the teacher as a facilitator in the construction of knowledge (Martínez, 2008, as cited in Bolaño Muñoz, 2020). Within this approach, learning occurs through the reformulation and restructuring of prior concepts, adapting them to new circumstances that foster the creation of new knowledge, as stated by Arteaga Martínez and Macías Sánchez (2016). Rooted in the work of Piaget and Vygotsky, the constructivist paradigm rests on the following principles: first, learning is grounded in the learner's active engagement; second, knowledge acquisition progresses through stages of equilibrium and disequilibrium in which prior understandings are challenged; and third, cognitive conflict among members of the same social group facilitates the construction of knowledge (Arteaga Martínez & Macías Sánchez, 2016).

The two foregoing approaches share a central premise: the learner must master the mathematical foundations. A third, more traditional approach within the teaching–learning process is empiricism. As Arteaga Martínez and Macías Sánchez (2016) note, in the empiricist approach knowledge is not contextualized because the student is considered unable to construct it; instead, the student learns what the teacher presents, and the instructor assumes the primary role in the process (p. 12).

More recently, competence-based teaching has gained increasing prominence. As defined by Cejas Martínez, Rueda Manzano, Cayo Lema, and Villa Andrade (2019), it is a teaching–learning process oriented toward the acquisition of skills, knowledge, and abilities—through procedures and attitudes—that enhance performance and enable individuals to meet organizational or institutional goals. This model continues to expand and is being widely implemented across educational institutions.

Although these educational perspectives share the common objective of improving students' mathematical learning, they differ substantially in their assumptions regarding how knowledge is acquired and how instruction should be organized. The idealist and empiricist perspectives primarily emphasize the

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transmission of mathematical knowledge, whereas constructivist and competency-based approaches place greater emphasis on students' active participation, reasoning, communication, and the application of knowledge in authentic contexts. Rather than considering these perspectives as mutually exclusive, the present study adopts an integrative standpoint that recognizes the importance of conceptual rigor while promoting active learning, mathematical modeling, and the meaningful use of technological resources to support the development of problem-solving competencies.

Against this backdrop of approaches to the teaching–learning process—not only in mathematics but also in other fields—educators face the challenge of teaching in a context of expanding access to technology. Technology is not at odds with these approaches; rather, it can be integrated into them in productive ways. In this regard, the following reflection is pertinent:

> The need to change the training paradigm in higher education so as to align it with countries' internal demands—and with the challenges posed to higher education institutions by globalization, the knowledge society, the cognitive revolution, and technological development—has become increasingly evident, particularly given the rise of ICTs across diverse domains of human activity. (Márquez González & Sánchez Leal, 2010, p. 1)

Higher education faces challenges comparable to those encountered by students upon entering university; accordingly, a viable response for instructors is to integrate technology into teaching. Today, computers are widely available as educational resources. At the tertiary level, technology integration—aimed at promoting pedagogical change—has shown favorable outcomes, reflected in students' enthusiasm for engaging with concepts and in the achievement of course objectives (Licona Meneses & Veytia Bucheli, 2020). As Salat (2009) notes, technology alters the very nature of mathematical activity; in practice, mathematics itself is reshaped by technology, in part because access to computers and specialized software across various areas is increasingly widespread.

A range of software tools currently supports mathematics teaching across subfields, including GeoGebra, Microsoft Mathematics, MATLAB, Maxima, wxMaxima, Wolfram Alpha, and Symbolab, among others. In the face of this technological expansion, the instructional task is not to restrict student access but to integrate these tools critically into classroom practice so that they support mathematical learning without displacing foundational understanding. As emphasized by Salat (2009), free and open-source software merits particular attention, since for many students it represents the only feasible means of access. Among the freely accessible and user-friendly options, GeoGebra stands out as a dynamic environment for teaching and learning mathematics—applicable to geometry, algebra, calculus, and statistics. In this context, certain technological tools—not only consistent with the pedagogical approaches discussed above but also complementary to them—offer multiple representations of mathematical objects (graphical, algebraic, statistical, and tabular), thereby enriching teaching and learning processes.

One notable advantage of these tools is the ability to work simultaneously with a graphical view—centered on geometric aspects—and an algebraic view that connects symbols and expressions with dynamic geometric representations. Although motivation can significantly influence learning even in non-technological contexts, the judicious use of technological resources can have an even greater impact when motivation alone is insufficient (Márquez González & Sánchez Leal, 2010). Leveraging such tools is therefore not a marginal option but an increasingly necessary strategy for strengthening instructional practice. In particular, open-source software provides a valuable pathway for deepening students' understanding of specific mathematical concepts.

With respect to problem solving, it is essential both to possess relevant knowledge and to apply it in real or familiar contexts. Prior knowledge can be mobilized to confront new situations; while errors may arise and require correction, the aim is to maximize students' problem-solving potential (Montero & Mahecha, 2020). A persistent challenge, however, is the lack of a clearly established teaching methodology for problem solving in mathematics education. Evidence from PISA suggests that low performance correlates with instruction centered on memorization (Donoso Osorio et al., 2020). In this context, many instructional resources reveal disconnections between prior and new knowledge, even when they attempt to situate tasks in familiar settings. Effective instruction in problem solving requires an analysis grounded in scientific inquiry that clarifies context: beginning with a careful reading, identifying key data, relating them to known or missing information, expressing them in mathematical terms, and ultimately reaching a solution (Cedeño Loor et al., 2019).

### 1.1 Traditional approaches to mathematical problem solving

In mathematics teaching, difficulties in problem solving often stem from traditional methodologies characterized by rigid roles, rote memorization, and repetitive drill. Such approaches are poorly aligned with meaningful learning: they tend to deliver prepackaged knowledge, assign students a passive role, and position the instructor at the center of a monotonous, transmission-oriented classroom. Attitudinal factors in the learning environment also play a significant role, as students' self-efficacy directly influences their motivation to learn (Moreta-Herrera et al., 2019).

Within problem solving, the *Pólya method* offers a prominent alternative. Recent studies suggest that techniques associated with Pólya outperform traditional methods by fostering skill development, critical thinking, and reasoning, thereby improving both problem-solving processes and outcomes among adolescents. As a four-phase strategy—understand the problem, devise a plan, carry out the plan, and look back—the Pólya method helps manage complexity and has been implemented in countries such as Peru, Mexico, Brazil, and Guatemala (Saucedo et al., 2019). When combined with technological tools and mathematical software, it supports effective strategies that promote sustained learning.

Pólya's approach is both reflective and procedural, involving an interactive sequence of steps. It integrates theory and practice, beginning with comprehension and concluding with a retrospective appraisal to identify consistencies or deficiencies in the process (Saucedo et al., 2019).

Another well-known approach is René Descartes' *Cartesian method*, which proposes decomposing a problem into simpler parts and applying logical deduction to reach a solution. In the *Discourse*—specifically its second part—Descartes outlines four steps that later informed the clinical method employed in medicine (Gómez Alcalá, 2020). Analysis, in this sense, is constructive: it reveals "the true way in which a thing has been methodically invented," yielding an analytic–constructive procedure. Geometric constructibility plays a central role: represent the problem with a diagram, posit the solution through geometric relations, and analyze these relations to obtain the resolution (Bello, 2021).

Integrating existing methods—particularly Pólya's framework together with ICT in higher-education courses such as differential calculus—provides instructors with tools to enrich their practice. Equally important is the formulation of problems in a clear and contextualized manner so that students perceive their relevance and applicability in everyday settings, thereby fostering an understanding of interconnectedness and the search for solutions (Munayco-Mesías et al., 2021).

From a methodological perspective, neither Pólya's heuristic framework nor Descartes' analytical method is sufficient on its own to address the diversity of optimization problems encountered in higher education. Pólya's approach provides a flexible structure that promotes reasoning, exploration, and decision making, whereas the Cartesian method contributes a systematic procedure for mathematical modeling and logical analysis. Consequently, the methodological guide proposed in this study integrates both perspectives by combining heuristic reasoning with formal mathematical analysis within a competency-based instructional framework supported by technological resources.

Regarding mathematical problem-solving competence, Meneses and Peñaloza (2019, as cited in Fernández Canoles, 2024) describe it as a set of abilities that enable students to analyze data, identify relevant information, formulate plans, and apply and test algorithms. In the same vein, this competence is evidenced when learners analyze data, select what is relevant, provide detailed explanations, plan, and correctly implement and validate algorithms (Meneses & Peñaloza, 2019, p. 12). For Cuesta, Aguiar, and Marchena (2015, as cited in Fernández Canoles, 2024), technological tools offer substantial advantages for instruction by enabling the personalization and adaptation of content and activities to students’ needs. More broadly, the incorporation of ICT in society—and especially in higher education—has grown markedly in recent years, becoming both a necessity and an essential resource for teaching and learning (Lay, Ortega, & Flores, 2023).

Developing mathematical reasoning competences for problem solving also depends on students’ interest and on instructors’ strategies (Fernández Canoles, 2024), underscoring the importance of integrating tools that promote innovation and sustained learning. To this end, accessible software such as GeoGebra, Microsoft Mathematics, wxMaxima, Wolfram Alpha, and Symbolab can be effectively leveraged. Most of these tools are freely available and provide substantial support for mathematical problem solving, particularly in optimization.

Optimization problems constitute a classical application of derivatives in differential calculus and can be efficiently addressed using the aforementioned tools.

### 1.2 Practical application of technologies to optimization

Multiple studies highlight the importance of problem analysis in the teaching and learning of calculus in general—and differential calculus in particular—including the study of functional variation and single-variable optimization. Despite a substantial body of literature, persistent challenges observed in both pre-university and university classrooms call for new strategies and didactic tools that directly address these issues and deepen students’ understanding (Mojica & Morales, 2020).

Converging findings point to weak conceptual understanding, procedural automatism, and insufficient command of differential-calculus content as major sources of difficulty in solving optimization problems. Reported obstacles include interpreting problem statements, selecting appropriate strategies and tools, and mastering the necessary mathematical content, such as domains of functions, reading and interpreting graphs, growth–decay analysis, and the use of first- and second-derivative criteria, among others (Sánchez et al., 2020; Portilla-Lara et al., 2019; Rodríguez, 2019; Williner et al., 2019; Morales et al., 2022; Cordero et al., 2019).

For single-variable optimization, several technological tools have proven effective; among them, *GeoGebra* stands out for its dynamic and visual interaction, which supports mathematical content across educational levels. When used heuristically, GeoGebra facilitates generalization by making behavioral patterns more apparent: through manipulation, experimentation, and theoretically grounded actions, students explore, formulate hypotheses, and develop solution strategies. In this context, the software's dynamic visualizations and interactive features function as heuristics with significant instructional impact on the teaching and learning of optimization problem solving (Morales et al., 2022).

Despite these advances, the existing literature continues to focus primarily on isolated instructional strategies, specific technological tools, or individual classroom experiences. Comparatively little attention has been given to the development of a comprehensive methodological framework that systematically guides the teaching, learning, and assessment of single-variable optimization problems while integrating technological resources in a meaningful and pedagogically coherent manner. Consequently, the central research question addressed in this study is the following: *How can a competency-based methodological guide be designed to systematically support the teaching, learning, and assessment of single-variable optimization problems through the meaningful integration of technological resources?* To address this question, this article proposes a five-phase methodological guide incorporating technological resources that combines mathematical modeling, problem-solving heuristics, competency-based assessment, and the critical use of freely available mathematical software, providing instructors with a coherent and transferable framework for teaching optimization in differential calculus.

## 2. METHODOLOGY

### 2.1 Study design

This study constitutes a theoretical–methodological (non-experimental) contribution. Its aim is to systematize and formalize a *guide for solving optimization problems in single-variable differential calculus*, aligned with competency development through problem solving and the transversal integration of ICT. No participants, empirical data collection, or inferential statistical analyses are involved. The guide is conceived as a *replicable and transferable* scaffold adaptable to diverse curricular contexts.

### 2.2 Materials and ICT tools

Freely accessible resources are employed for visualization, symbolic computation, and verification, including GeoGebra, Desmos, Wolfram Alpha, wxMaxima, and Symbolab. ICT is integrated for formative purposes (exploration, contrast/triangulation, and documentation) without supplanting mathematical reasoning.

### 2.3 Procedure for constructing the guide

The guide is organized into five clearly defined phases, which were also followed in the preparation of this article:

1. **Problem Comprehension:** close reading, extraction of data and conditions, diagram construction, and the establishment of a glossary of variables;
2. **Mathematical Formulation of the Problem:** definition of variables and parameters, formulation of constraints and domain, and construction of the objective function;

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3. **Strategy Design:** selection of criteria (Fermat, first- and second-derivative tests), identification of critical and singular points, and consideration of domain endpoints;

4. **Implementation and Results:** systematic execution of the plan, classification of extrema, and preparation of tables and representations;

5. **Verification and Problem Resolution:** algebraic and geometric checks, unit consistency, local sensitivity analysis, and final synthesis.

For each phase, we specify *inputs* (data, assumptions), *processes* (techniques and criteria), and *outputs* (evidence: equations, tables, figures), enabling instructors to implement *rubrics* and provide formative feedback.

### 2.4 Evaluation criteria and indicators of achievement

The *indicators of achievement* are aligned with four competency domains:

- **Modeling:** well-defined variables and parameters; consistent constraints and domain; explicit assumptions;
- **Reasoning and argumentation:** appropriate selection of criteria; construction and explanation of sign charts; justification of the optimum;
- **Mathematical communication:** consistent notation; cross-referencing of equations and figures; correct units; expository clarity;
- **Critical use of ICT:** reproducible exploration and verification; inclusion of screenshots or source files when appropriate; independence of results from the chosen tool.

We recommend assessing performance using levels (incipient, basic, competent, advanced) and documenting concrete evidence at each phase.

### 2.5 Scope and limitations

This contribution is methodological in nature and does not include empirical validation of the guide in classroom settings. It is presented as a *replicable* basis

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for intervention designs, case studies, and impact evaluations across diverse curricular contexts.

# *3. RESULTS*

## 3.1 Methodological Guide

Building on the integration of Pólya’s method and Descartes’ Cartesian method, we propose the following steps for solving optimization problems.

### 3.1.1 1. Problem Comprehension

*Guiding question:* What is recommended to understand the problem?

*Suggested actions:*

- Read the problem carefully to form an overall understanding of the situation;
- read it again in detail, identifying keywords that describe the core mathematical context;
- clearly distinguish between the information provided (given data) and the information requested (unknowns to be determined);
- construct a representative sketch or diagram, labeling known and unknown quantities and variables.

*Indicators of achievement:*

- Correctly identifies the overall situation described in the problem;
- highlights keywords and relevant data in the statement;
- clearly differentiates between given data and required data;
- presents a diagram or visual representation with clearly labeled quantities and variables.

### 3.1.2 2. Mathematical Formulation of the Problem

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*Guiding question:* What is recommended to formulate the problem mathematically?

*Suggested actions:*

• Assign variables to both the given quantities and those to be determined;

• place and label these variables on the sketch, figure, or diagram developed in the previous phase;

• determine the domain of the variables so that the problem is meaningful within its context (this may require background knowledge from the application domain);

• establish the law, equation, or principle that relates the given variables;

• derive a function that links the known variables to the quantity to be determined;

• impose conditions or constraints that reduce the relation to two variables only: one independent and one dependent (the present guide focuses on real-valued functions of a single real variable);

• express the objective function in terms of the variable to be optimized;

• test sample values to confirm that the model behaves correctly within the domain, and verify that outside the constrained domain the model does not apply.

*Indicators of achievement:*

• Correctly assigns variables to both given and required quantities;

• clearly represents the variables in the diagram or figure;

• defines an appropriate domain for the variables, consistent with the problem context;

• correctly identifies the law or principle that relates the variables;

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• derives the function describing the relationship between the given and target variables;

• reduces the relationship to a single independent variable and one dependent variable;

• correctly expresses the objective function;

• validates the model by evaluating test values both within and outside the constrained domain.

#### 3.1.3 3. Strategy Design

*Guiding question:* What is recommended to design a strategy to solve the problem?

*Suggested actions:*

• Once the objective function has been established, recognize that the task becomes a calculus problem involving the identification of extreme values;

• recall and apply the standard procedures for determining extrema of real-valued functions of a real variable;

• apply differential-calculus theorems and tests to identify maxima and minima (e.g., Fermat’s theorem; first-derivative test; second-derivative test);

• analyze the domain conditions of the function and identify possible critical points;

• incorporate basic mathematical software (GeoGebra, Desmos, Microsoft Mathematics, Wolfram Alpha, Symbolab, among others) to:

– visualize the function and its behavior;

– verify critical points and extrema;

– support algebraic and differential computations;

• keep in mind that digital tools can be integrated throughout the process—even from the first phase—provided that the objectives and competencies are clearly defined.

*Indicators of achievement:*

• Recognizes that the task is an optimization problem that requires the use of differential calculus;

• correctly applies the procedures for determining extreme values;

• appropriately applies theorems and tests to identify maxima and minima;

• identifies and analyzes the critical points of the objective function;

• integrates mathematical software for visualization, computation, and verification;

• understands the transversal role of ICT throughout the solution process;

• demonstrates clarity regarding the objectives and competencies associated with this phase.

### 3.1.4 4. Strategy Implementation and Results

*Guiding question:* How is the strategy implemented to obtain the problem's results?

*Suggested actions:*

• Systematically implement the procedures established in the chosen strategy;

• compute the first derivative $f'(x)$ of the objective function;

• identify all critical candidates, including:

– stationary points: $x \in D(f)$ and $f'(x) = 0$;

– singular points: where $f'(x)$ does not exist but $f(x)$ is defined;

– endpoints of the domain (when the context or constraints impose bounds);

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• perform a monotonicity analysis (first-derivative test):

– construct a sign chart for $f'(x)$ over the intervals determined by the critical candidates and domain endpoints;

– determine where $f'(x) > 0$ (increasing) and $f'(x) < 0$ (decreasing);

• perform a concavity and inflection analysis (second-derivative test, if necessary):

– compute the second derivative $f''(x)$;

– determine concavity intervals ($f''(x) > 0$: concave up; $f''(x) < 0$: concave down);

– identify inflection points by confirming sign changes of $f''(x)$;

• classify extrema:

– apply the first-derivative test (sign changes in $f'(x)$) to classify maxima and minima;

– when conclusive, apply the second-derivative test to confirm the nature of the extrema;

• compare the values of the objective function at all candidates (critical points and domain endpoints) to determine the optimal value;

• plot the objective function and highlight the relevant values;

• integrate technological tools (GeoGebra, Wolfram Alpha, wxMaxima, Symbolab) to verify derivatives, solve equations, and visualize the function's behavior.

*Indicators of achievement:*

• Correctly identifies all critical candidates, including singular points and domain endpoints;

• constructs a sign chart for $f'(x)$ and correctly determines the intervals of increase and decrease;

• determines the function’s concavity via f″(x) and identifies inflection points when present;

• correctly applies the first- and second-derivative tests to classify critical points;

• uses digital tools to confirm computations, verify results, and visualize behavior.

#### 3.1.5 5. Verification and Problem Resolution

*Guiding question:* How can one verify that the results are correct and consistent with the stated problem?

*Suggested actions:*

• Review all stages of the solution process to ensure that no conceptual or operational errors have been made;

• verify the results mathematically, checking that they satisfy the equations, conditions, and constraints originally posed;

• assess the logical coherence of the solution within the problem’s context, confirming that the results make sense physically, geometrically, economically, or otherwise, as appropriate;

• revisit the mathematical model to test the validity of the hypotheses and simplifications adopted;

• confirm correspondence with the real situation, ensuring that the obtained values could occur in the described scenario;

• compare the results with graphical representations and/or simulations produced by mathematical software;

• analyze the stability of the solution under small perturbations of the initial data;

• present the final solution clearly in terms of the original variables and with the appropriate units.

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*Indicators of achievement:*

- Confirms the mathematical validity of the obtained results;
- verifies the coherence of the solution with the context and stated conditions;
- recognizes the correspondence between the mathematical model and the real situation;
- cross-checks the results against graphical representations and software-based simulations;
- evaluates the robustness of the solution under variations in the initial data;
- communicates the final solution clearly and precisely, using correct units.

### 3.2 Application of the Methodological Guide to an Example Problem

#### 3.2.1 Example Problem

A rectangular yard is to be fenced using the wall of a house that is 40 ft wide. A total of 160 ft of fencing is available. Describe how the fencing should be used to enclose the largest possible area *(Zill & Wright, 2011, p. 241, problem 18)*. Figure 1 illustrates the situation.

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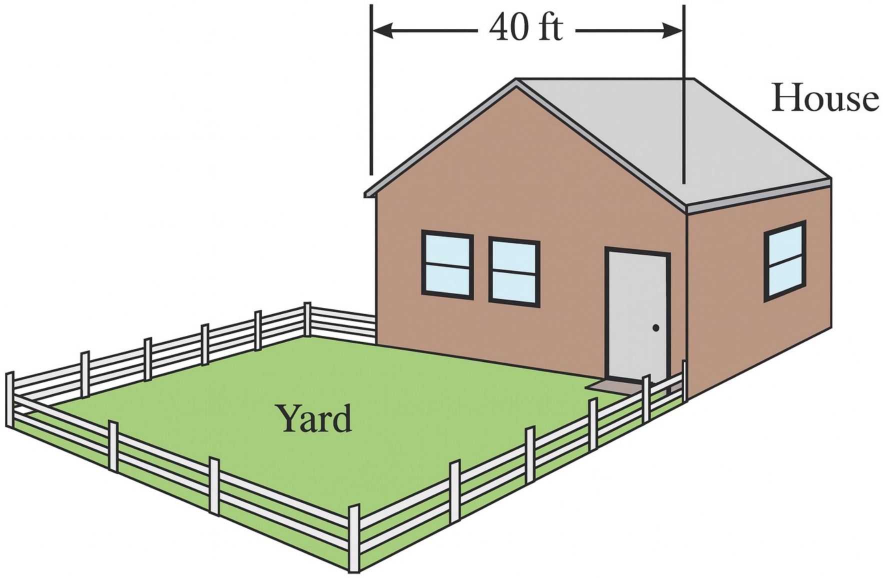
**Figure 1. Rectangular yard attached to the house, with a width of 40 ft.**



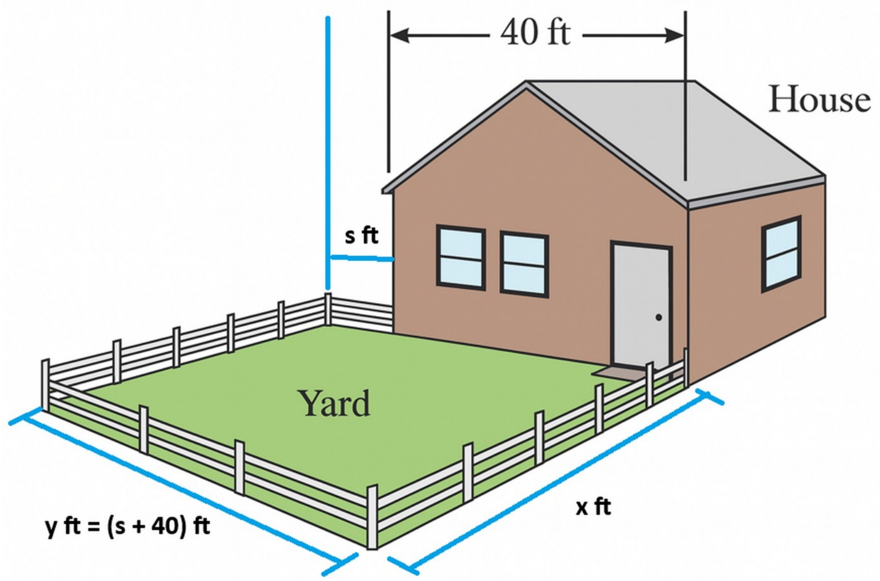

**Note.** Adapted from Zill & Wright (2011, p. 241).

#### 3.2.2 1. Problem Comprehension

*Problem statement.* We aim to fence a *rectangular* yard attached to a *house* whose frontage provides a fixed side of 40 ft. Only *160 ft* of fencing is available for the remaining sides. The objective is to determine how to allocate the fencing to maximize the enclosed *area*.

*Variables.* Let x denote the length *perpendicular* to the house (in ft), y the length *parallel* to the house on the opposite side, and s the additional short segment along the house that completes the front (so that y = 40 + s).

**Figure 2. Problem diagram showing the variables x, y, and s.**




**Note.** Adapted from Zill & Wright (2011, p. 241).

*Context.* The house provides one fixed side (40 ft), so only three fence segments are required: two of length x, one opposite the house of length y, and a short completion segment s adjacent to the house. This modifies the usual perimeter relation and determines the formulation of the objective function.

#### 3.2.3 2. Mathematical Formulation

The diagram in Figure 2 guides the modeling. The variables are

$$\begin{aligned} x &= \text{perpendicular length (ft)}, \\ y &= \text{parallel length opposite the house (ft)}, \\ s &= \text{additional segment along the house (ft)}. \end{aligned} \tag{1}$$

Because the house covers 40 ft of the front, the parallel side satisfies

$$y = s + 40. \tag{2}$$

The available fencing (160 ft) is used for the two sides of length x, the opposite side y, and the completion segment s, which yields the linear constraint

$$2x + y + s = 160. \tag{3}$$

Combining (2) with (3) gives 2x + 2s + 40 = 160, hence x + s = 60 and, therefore,

$$s = 60 - x. \tag{4}$$

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The area of the rectangle is

$$A(x, y) = x\,y. \tag{5}$$

Using y = s + 40 and (4), we obtain a single-variable objective function,

$$A(x) = x\big[(60 - x) + 40\big] = x(100 - x),$$

so the function to be optimized is

$$A(x) = x(100 - x). \tag{6}$$

Feasibility requires positive lengths $x > 0$ and $s > 0$. Using (4), this becomes

$$x > 0 \quad \text{and} \quad 60 - x > 0, \tag{7}$$

hence the working domain

$$0 < x < 60. \tag{8}$$

We thus arrive at the single-variable optimization problem

$$\max_{0<x<60} A(x) = x(100 - x), \tag{9}$$

a classical one-dimensional calculus problem with domain (8).

#### 3.2.4 3. Solution Strategy

Given (6) on (8), the problem is addressed using differential calculus: (i) identify critical candidates by solving $A'(x) = 0$ within the domain; (ii) check for singular points (none arise here, since A is a polynomial); (iii) consider the domain endpoints; and (iv) classify candidates using first- and second-derivative tests. Graphical checks (GeoGebra, Wolfram Alpha, Desmos) support verification.

#### 3.2.5 4. Implementation and Results

**Derivative and critical point.** The first derivative is $A'(x) = 100 - 2x$. Solving $A'(x) = 0$ yields $x* = 50$, which lies in (8).

**No singular points.** Since A is a quadratic polynomial, $A'$ exists for all $x \in \mathbb{R}$.

**Endpoints.** Evaluating at the domain boundaries gives $A(0) = 0$ and $A(60) = 60(100 - 60) = 2400$. As shown below, the maximum occurs in the interior.

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**First-derivative test, monotonicity.** On (0, 50), $A'(x) > 0$ (increasing); at $x = 50$, $A'(x) = 0$; on (50, 60), $A'(x) < 0$ (decreasing). This indicates a local maximum at $x = 50$, as shown in Table 1.

**Table 1. Sign chart of $A'(x)$ and behavior of $A(x)$.**

| Interval | Sign of $A'(x)$ | Behavior of $A(x)$ | Note |
|---|---|---|---|
| $0 < x < 50$ | + | Increasing | Approaches the maximum |
| $x = 50$ | 0 | — | Critical point |
| $50 < x < 60$ | − | Decreasing | Recedes from the maximum |

**Second-derivative test, concavity.** We have $A''(x) = -2 < 0$ for all x, so A is concave down everywhere; therefore, $x* = 50$ is an *absolute* maximizer on (0, 60).

**Optimal dimensions and maximum area.** From (4)–(2), $s* = 60 - 50 = 10$ and $y* = s* + 40 = 50$. Thus $x* = 50$ ft, $y* = 50$ ft, $s* = 10$ ft, and the maximum area is

$$A_{\max} = A(50) = 50(100 - 50) = 2500 \text{ ft}^2.$$

**Technology support.**

**GeoGebra: plotting and verification.** *Purpose.* Visualize $A(x) = x(100 - x)$ on $0 < x < 60$, locate the critical point, and confirm the absolute maximum using derivative tests and concavity.

*Operational steps in* **GeoGebra** *(commands).*

1. A(x) := x∗(100 - x)

2. f := Function(A, 0, 60)

3. A'(x) ⇒ 100 − 2x

4. Solve(A'(x) = 0) ⇒ {x = 50}

5. P := (50, A(50)) ⇒ P = (50, 2500)

6. Extremum(A) ⇒ (50, 2500) *(automatic alternative)*

7. A''(x) ⇒ −2 *(negative concavity over the entire domain)*

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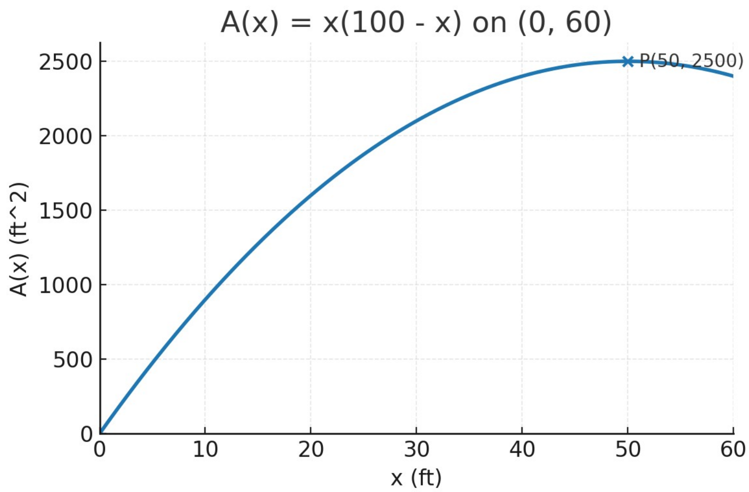
8. Tangent(P, f) ⇒ horizontal tangent at y = 2500

9. SetLabelMode(P, 1) ⇒ display the label of P

*Analysis.* The restricted plot f = Function(A, 0, 60) is a downward-opening parabola. Solve[A'(x)=0] yields $x* = 50$; the point P = (50, 2500) is the vertex. Since $A''(x) = -2 < 0$ everywhere, $x*$ is an *absolute* maximizer. The monotonicity behavior is consistent with the first-derivative sign chart: increasing on (0, 50) and decreasing on (50, 60). Hence, the optimal dimensions are $x* = 50$ ft, $y* = 50$ ft, and $s* = 10$ ft, with $A_{max} = 2500$ ft² (see Figure 3).

**Figure 3. GeoGebra output for A(x) = x(100 − x) on (0, 60), showing the vertex P(50, 2500) and the negative concavity (A″(x) = −2).**




**wxMaxima: automation of derivatives and plotting.** The following *wxMaxima* script (Maxima) (1) defines A(x) = x(100 − x), (2) computes A′(x), A″(x), and the interior critical point, and (3) generates the plot on (0, 60), highlighting the absolute maximum P(50, 2500) (see Figure 4).

**Listing 1.** wxMaxima script to confirm the optimum and plot A(x) = x(100 − x) on (0, 60)

```
/* Define the area function and its first and second derivatives */
A(x) := x*(100 - x)$
A1    : diff(A(x), x)$
A2    : diff(A(x), x, 2)$

/* Solve A'(x)=0 and obtain the interior critical point */
crit  : solve(A1 = 0, x)$
xstar : rhs(first(crit))$
Amax  : ev(A(x), x = xstar)$

/* Evaluate A(x) at the domain endpoints for contrast */
A0  : ev(A(x), x = 0)$
```

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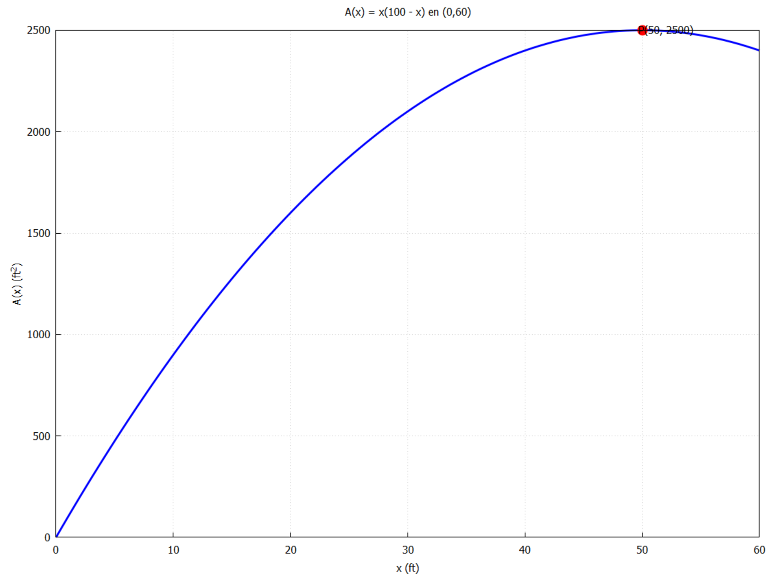
```
A60 : ev(A(x), x = 60)$

/* Display the curve A(x) on (0,60) and mark the maximum */
wxdraw2d(
  xlabel = "x (ft)", ylabel = "A(x) (ft^2)",
  grid = true,
  color = blue, line_width = 3,
  explicit(A(x), x, 0, 60),
  color = red, point_type = filled_circle, point_size = 2.5,
  points([[xstar, Amax]]),
  color = black, label(["P(50, 2500)", xstar + 2, Amax]),
  title = "A(x) = x(100 - x) on (0,60)"
)$

/* Export the same plot to PNG (saved as wxMaxima.png) */
draw2d(
  terminal   = pngcairo,
  file_name  = "wxMaxima",
  dimensions = [1800, 1100],
  xlabel = "x (ft)", ylabel = "A(x) (ft^2)",
  grid = true,
  color = blue, line_width = 3,
  explicit(A(x), x, 0, 60),
  color = red, point_type = filled_circle, point_size = 2.5,
  points([[xstar, Amax]]),
  color = black, label(["P(50, 2500)", xstar + 2, Amax]),
  title = "A(x) = x(100 - x) on (0,60)"
)$
```

**Figure 4. wxMaxima output for A(x) = x(100 − x) on (0, 60), highlighting the maximum P(50, 2500).**




**Other open-source and freely available tools, not used here.** All the following tools are open-source or freely available. Depending on instructional goals and the *type of problem* (symbolic vs. numerical analysis; single- vs. multivariable; need for interactivity or reproducibility), one may integrate tools such as *SageMath* (full CAS), *GNU Octave* (numerical optimization; MATLAB-like syntax), *Scilab* (numerics and visualization), *Python* with SymPy/SciPy (symbolic differentiation and optimization) and *Jupyter* notebooks for traceability, *Julia* with Optim.jl and SymPy.jl, or *R* with Ryacas/Deriv for basic symbolic support. Selection should be guided by learning outcomes, problem characteristics, and resource availability.

#### 3.2.6 5. Verification and Problem Resolution

With the optimal values obtained in Phase 4, the solution is verified both mathematically and contextually to ensure consistency with the model and its constraints.

**Algebraic verification of the constraints.** Let $x* = 50$. From (4), $s* = 60 - x* = 10$; and from (2), $y* = s* + 40 = 50$. Substituting into the linear constraint (3) yields

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$$2x^* + y^* + s^* \;=\; 2(50) + 50 + 10 \;=\; 160,$$

so the total available fencing is used exactly. The physical conditions (7) and the domain (8) are also satisfied, since $0 < x* < 60$ and $s* > 0$.

**Consistency with the objective function.** Evaluating the area at the optimum using (6) gives $A_{max} = A(50) = 2500$ ft², which is consistent with the result obtained in Phase 4.

**Analytical confirmation of optimality.** Differential criteria confirm that (i) the sign change of $A'(x)$ around $x*$ (Table 1) indicates a local maximum, and (ii) $A''(x) < 0$ over the entire domain, ensuring concavity downward and, therefore, that $x*$ is an *absolute* maximum on (0, 60).

**Local contrast with nearby values.** As a practical verification, compare $A(x)$ at values near $x*$, as shown in Table 2:

**Table 2. Local numerical check around $x* = 50$.**

| x | s = 60 − x | y = s + 40 | A(x) = x(100 − x) |
|---|---|---|---|
| 49 | 11 | 51 | 49 · 51 = 2499 |
| 50 | 10 | 50 | 50 · 50 = 2500 |
| 51 | 9 | 49 | 51 · 49 = 2499 |

The symmetry $A(49) = A(51)$ and the maximum at $x*$ confirm optimality.

**Geometric and unit consistency.** The dimensions $(x*, y*, s*) = (50, 50, 10)$ define a feasible enclosure (all positive) consistent with the geometry. Lengths are expressed in feet (ft) and area in square feet (ft²), in accordance with (5).

**Robustness and local sensitivity.** The loss of area away from the optimum follows from the *second-order Taylor theorem* and, in this problem, coincides exactly with the *vertex form* of the parabola.

**(i) Taylor theorem, second order.** Let $x*$ be the interior maximizer and $h := \Delta x$ a small perturbation. For $A \in C^2$ near $x*$ (see, e.g., Apostol, 1974, ch. 7; Rudin, 1976, ch. 5),

$$A(x^* + h) = A(x^*) + A'(x^*)\,h + \tfrac{1}{2}A''(x^*)\,h^2 + R_2(h), \qquad \lim_{h\to 0} \frac{R_2(h)}{h^2} = 0. \qquad (10)$$

Here $A(x) = x(100 - x)$ is quadratic, so $A'''(x) \equiv 0$ and $R_2(h) \equiv 0$. Moreover, $A'(x*) = 0$ and $A''(x) \equiv -2$. Substituting into (10) gives

$$A(x^* + h) - A(x^*) = \tfrac{1}{2}A''(x^*)\,h^2 = -\,h^2. \tag{11}$$

Hence, from (11), the *area loss* relative to the maximum is exactly

$$\Delta A := A(x^*) - A(x^* + h) = h^2 = (\Delta x)^2. \tag{12}$$

*Units.* Since A is measured in $ft^2$ and x in ft, $A'(x)$ has units of ft and $A''(x)$ is dimensionless; thus $|A''(x*)|/2 = 1$ maps $(\Delta x)^2$ ($ft^2$) directly to area loss ($ft^2$).

**(ii) Vertex form, completing the square.** From (6) we obtain $A(x) = -(x - 50)^2 + 2500$, and therefore $A(50) - A(x) = (x - 50)^2 = (\Delta x)^2$, which matches (12).

**Interpretation.**

- *Quadratic penalty:* the loss increases proportionally to $(\Delta x)^2$; doubling the deviation from $x*$ quadruples the area loss.
- *Numerical example:* with $x* = 50$, moving to $x = 49$ or $x = 51$ gives $\Delta x = 1$ ft and, by (12), $\Delta A = 1$ $ft^2$, consistent with Table 2.
- *Curvature and "flatness" at the top:* in general, $A(x*) - A(x* + h) = (|A''(x*)|/2)\, h^2 + o(h^2)$. Smaller $|A''(x*)|$ implies a flatter peak and greater robustness (lower sensitivity) to small variations in x. Here $|A''(x*)| = 2$, so the local sensitivity coefficient is exactly $|A''(x*)|/2 = 1$.

**Verification (summary).** Verification should revisit *all stages of the model* and its internal consistency: careful reading and interpretation of data, definition of variables and domain, formulation of constraints and the objective function, differentiation, classification of candidates, and consistency of units. In addition, a *software cross-check* (e.g., GeoGebra, Desmos, Wolfram Alpha, wxMaxima, Symbolab) is advisable to plot $A(x)$ on $(0, 60)$, confirm $A'(50) = 0$ and $A''(x) = -2 < 0$, and visualize the maximum at $x = 50$.

**Contextual validity.** Verification is not purely mathematical: the solution must also be *validated against real conditions* (geometric and physical feasibility,

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positivity of lengths, full use of fencing, correct units, and interpretability within the context).

**Problem solution.** Under (3) and (8), the area-maximizing configuration is

$$x^* = 50 \text{ ft}, \qquad y^* = 50 \text{ ft}, \qquad s^* = 10 \text{ ft}, \qquad A_{\max} = 2500 \text{ ft}^2,$$

which satisfies (2) and (4) and is optimal on the specified domain.

**Methodological note.** Although in this case one could *infer by inspection* (the vertex of a downward-opening parabola) that the maximum occurs at x = 50, we rigorously followed *all five phases of the guide*. The objective was not merely to solve the problem, but to *demonstrate the procedure and the underlying reasoning* so that it can be transferred to analogous situations.

## 4. DISCUSSION

The methodological guide incorporating technological resources proposed in this study provides a structured framework for teaching, learning, and assessing optimization problems in single-variable differential calculus through the integration of mathematical modeling, problem-solving heuristics, and competency-based instruction. Unlike approaches that focus primarily on the use of a specific technological tool or on isolated classroom experiences, the present proposal emphasizes the systematic organization of the complete problem-solving process through five sequential phases supported by observable criteria and indicators of achievement.

From a theoretical perspective, the proposed guide is consistent with the educational foundations discussed in the Introduction. It combines the heuristic reasoning promoted by Pólya, the systematic mathematical analysis associated with the Cartesian method, and competency-based educational approaches that emphasize mathematical reasoning, communication, and the meaningful integration of ICT. Rather than replacing mathematical reasoning, technological resources are incorporated as complementary tools for visualization, verification, and exploration, thereby reinforcing conceptual understanding.

This contribution should therefore be interpreted as a theoretical–methodological framework rather than as an empirically validated instructional intervention. Consequently, the objective of this article is not to measure learning outcomes or instructional effectiveness, but to provide a coherent and transferable methodological proposal that can guide teaching practice and serve as a foundation for formative assessment.

Accordingly, a natural direction for future research is to begin by validating the proposed guide through expert judgment and subsequently implementing it in university differential calculus courses. Such studies may employ quasi-experimental or mixed-methods research designs to evaluate the guide's impact on students' mathematical reasoning, problem-solving competencies, and the meaningful integration of technological resources into the teaching and learning process.

## *5. CONCLUSIONS*

This article presents a theoretical–methodological guide for solving optimization problems in single-variable differential calculus. Its main contribution is a five-phase framework with explicit inputs, processes, and outputs, together with observable indicators of achievement aligned with competency development (modeling, reasoning, communication, and the critical use of ICT). The proposed framework provides a coherent structure for organizing the teaching, learning, and assessment of optimization problems while incorporating technological resources in a meaningful and pedagogically consistent manner. It also establishes a rigorous theoretical basis for the design of rubrics and protocols for formative assessment. As future work, we envisage empirical validation in differential calculus courses and student cohorts, as well as the extension of the proposed framework to constrained and multivariable optimization problems.

## *AUTHOR CONTRIBUTIONS*

All authors contributed equally to the overall development of the manuscript. The order of authorship reflects the priority of leadership and responsibility: Yahaira Antonia Rodríguez Hernández, Juan Toribio Milane, and Cándido Benito Silverio Llanos. The distribution of tasks is as follows:

- *Conceptualization:* Yahaira Antonia Rodríguez Hernández (lead); Juan Toribio Milane (co-lead); Cándido Benito Silverio Llanos (support).
- *Methodology:* Yahaira Antonia Rodríguez Hernández (lead); Juan Toribio Milane (co-lead); Cándido Benito Silverio Llanos (support).
- *Formal analysis:* Yahaira Antonia Rodríguez Hernández (lead); Juan Toribio Milane (support); Cándido Benito Silverio Llanos (support).
- *Writing–original draft (Introduction):* Cándido Benito Silverio Llanos (lead); Yahaira Antonia Rodríguez Hernández (co-lead); Juan Toribio Milane (support).
- *Writing–original draft (remainder of the manuscript):* Yahaira Antonia Rodríguez Hernández (lead); Juan Toribio Milane (support); Cándido Benito Silverio Llanos (support).
- *Writing–translation (English version):* Juan Toribio Milane (lead).
- *Writing–review & editing:* All authors (equal contribution).
- *Visualization and software (ICT support):* Juan Toribio Milane (lead); Yahaira Antonia Rodríguez Hernández (co-lead); Cándido Benito Silverio Llanos (support).
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## *CONFLICT OF INTEREST*

The authors declare that there are no conflicts of interest.

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